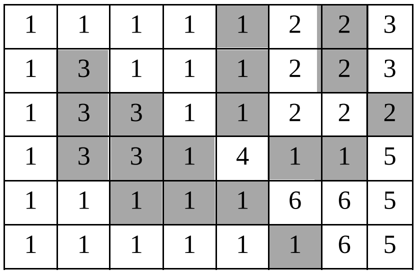
I.



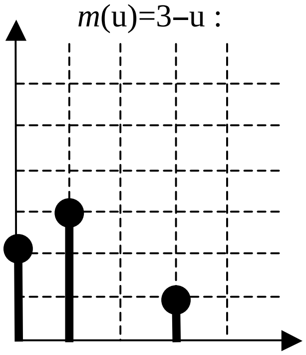
1	1	1	1	1	2	2	3
1	3	1	1	1	2	2	3
1	3	3	1	1	2	2	2
1	3	3	1	4	1	1	5
1	1	1	1	1	6	6	5
1	1	1	1	1	1	6	5

There is only one 8-connected component.
The equivalence label class is: $\overline{1}=\overline{2}=\overline{3}=\{1,2,3\}$

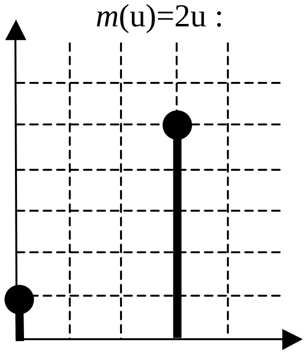
There are five 4-connected components.
The equivalence label classes are: $\overline{1}=\{1\}$, $\overline{2}=\{2\}$, $\overline{3}=\{3\}$, $\overline{4}=\{4\}$, $\overline{5}=\overline{6}=\{5,6\}$

II.

m(u)=3–u :



m(u)=2u :



III.

As seen in class: $$(f * g)(x,y) = \sum_{i=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} f(i,j)\, g(x-i,y-j).$$

Now, consider convolution at the pixel whose grey level is 1, i.e., at the pixel of coordinates (0,0).

We have: $$\begin{aligned}(f * g)(0,0) &= \sum_{i=-1}^{1} \sum_{j=-1}^{1} f(i,j)\, g(-i,-j)\\ &= f(-1,-1)g(1,1) + f(-1,0)g(1,0) + f(-1,1)g(1,-1) +\\ &\quad f(0,-1)g(0,1) + f(0,0)g(0,0) + f(0,1)g(0,-1) +\\ &\quad f(1,-1)g(-1,1) + f(1,0)g(-1,0) + f(1,1)g(-1,-1)\\ &= a_{11}\, g(1,1) + a_{12}\, g(1,0) + a_{13}\, g(1,-1) +\\ &\quad a_{21}\, g(0,1) + a_{22}\, g(0,0) + a_{23}\, g(0,-1) +\\ &\quad a_{31}\, g(-1,1) + a_{32}\, g(-1,0) + a_{33}\, g(-1,-1)\end{aligned}$$

If we assume that the values g(1,−1), g(0,−1), g(−1,1), g(−1,0) and g(−1,−1) are 0, we get: $(f * g)(0,0) = 12a_{11} + 11a_{12} + 2a_{21} + a_{22}$. For a smoother and more natural transition, we may also assume that the first row and column repeat themselves outside the M×N rectangular region (reflected indexing). We then get: $(f * g)(0,0) = 12a_{11} + 11a_{12} + 11a_{13} + 2a_{21} + a_{22} + a_{23} + 2a_{31} + a_{32} + a_{33}$.

First alternative

0	0	0	0
0	1	2	3
0	11	12	13
0	21	22	23

Second alternative

1	1	2	3
1	1	2	3
11	11	12	13
21	21	22	23

IV.

Let (x,y) be an element of $\mathbb{Z}^2$, let f be an element of $\mathcal{M}^{\square}$ and g be an element of $\mathcal{M}$.

Can we define (f *g)(x,y) as $\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f(i,j)\, g(x-i,y-j)$?

Since f and g are mappings, every term f(i,j) g(x−i,y−j) in the double sum is defined. However, adding an infinite number of values might not make any sense. It makes sense here, and here is the reason. Assume f belongs to $\mathcal{M}^{\square}$. We can find two odd and positive integer values m and n such that for any (i,j) in $\mathbb{Z}^2$, if $i<-(m-1)/2$ or $i>(m-1)/2$ or $j<-(n-1)/2$ or $j>(n-1)/2$ then $f(i,j)=0$. In other words, there exists an m×n rectangular region outside of which f is zero.

$$\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f(i,j)\, g(x-i,y-j) \text{ is therefore equal to the finite sum } \sum_{i=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} f(i,j)\, g(x-i,y-j).$$

The same is true if g belongs to $\mathcal{M}^{\square}$ (and f to $\mathcal{M}$). We can still find two integers m and n such that:

$$\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f(i,j)\, g(x-i,y-j) \text{ is equal to } \sum_{i=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} f(i,j)\, g(x-i,y-j).$$

What all this means is that convolution can be defined in a more general setting, where there is no need to distinguish between the left operand (the “convolution kernel”) and the right operand (the “image”).

1/ a) Here, both f and g represent elements of $\mathcal{M}^{\square}$. To compute f *g, you can consider that g is a very small “image”, with zero grey level values outside the 1×3 rectangular region.

We get: $f * g = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$.

“Image” g

0	0	0	0	0
0	0	0	0	0
0	−1	0	1	0
0	0	0	0	0
0	0	0	0	0

f * g

−1	0	1
−2	0	2
−1	0	1

For instance, $(f*g)(-1,0) = \sum_{i=-1}^{1}\sum_{j=-0}^{0} f(i,j)\, g(-1-i,-j) = \sum_{i=-1}^{1} f(i,0)\, g(-1-i,0) = 1\times(-1)+2\times 0+1\times 0 = -1$

and $(f*g)(0,0) = \sum_{i=-1}^{1}\sum_{j=-0}^{0} f(i,j)\, g(-i,-j) = \sum_{i=-1}^{1} f(i,0)\, g(-i,0) = 1\times 0+2\times(-1)+1\times 0 = -2.$

For g * f we also get $\begin{bmatrix}-1 & 0 & 1\\ -2 & 0 & 2\\ -1 & 0 & 1\end{bmatrix}$.

“Image” f

0	0	0	0	0
0	0	1	0	0
0	0	2	0	0
0	0	1	0	0
0	0	0	0	0

g * f

−1	0	1
−2	0	2
−1	0	1

In the end: f *g = g * f = fg.

b) f could be used as a convolution kernel for smoothing an image (a weighted average of grey level values is computed in a 3×1 neighbourhood). g could be used as a convolution kernel for detecting vertical edges. f *g = g * f = fg could be used for the same purpose.

c) $k(x,y) = (g*h)(x,y) = \sum_{i=0}^{0}\sum_{j=-1}^{1} g(i,j)\, h(x-i,y-j) = \sum_{j=-1}^{1} g(0,j)\, h(x,y-j)$

$$= g(0,-1)\, h(x,y+1) + g(0,0)\, h(x,y) + g(0,1)\, h(x,y-1) = -h(x,y+1) + h(x,y-1)$$

$$(f*(g*h))(x,y) = (f*k)(x,y) = \sum_{i=-1}^{1}\sum_{j=-0}^{0} f(i,j)\, k(x-i,y-j) = \sum_{i=-1}^{1} f(i,0)\, k(x-i,y)$$

$$= f(-1,0)\, k(x+1,y) + f(0,0)\, k(x,y) + f(1,0)\, k(x-1,y) = k(x+1,y) + 2k(x,y) + k(x-1,y)$$

$$\underset{\textit{of } k(x,y) \textit{ above}}{\overset{\textit{see expression}}{=}} [-h(x+1,y+1) + h(x+1,y-1)] + 2[-h(x,y+1) + h(x,y-1)] + [-h(x-1,y+1) + h(x-1,y-1)]$$

$$= -h(x+1,y+1) + h(x+1,y-1) - 2h(x,y+1) + 2h(x,y-1) - h(x-1,y+1) + h(x-1,y-1)$$

$$\underset{\textit{convolution}}{\overset{\textit{definition of}}{=}} \left(\begin{bmatrix}-1 & 0 & 1\\ -2 & 0 & 2\\ -1 & 0 & 1\end{bmatrix} * h\right)(x,y) \overset{\textbf{1/ a)}}{=} ((f*g)*h)(x,y)$$

We have shown that f *(g*h) = (f *g)*h.

2/ a) $(f*g)(x,y) = \sum_{i=-\frac{m-1}{2}}^{\frac{m-1}{2}}\sum_{j=-0}^{0} f(i,j)\, g(x-i,y-j) = \sum_{i=-\frac{m-1}{2}}^{\frac{m-1}{2}} f(i,0)\, g(x-i,y) = \sum_{i=-\infty}^{\infty} f(i,0)\, g(x-i,y)$

The finite sum can be replaced by the infinite sum because f(i,0) is 0 anyway if i is not between −(m−1)/2 and (m−1)/2 (we are “outside” the “convolution kernel” f). Moreover, if i≠x then g(x−i,y)=0 (because then we are “outside” the “image” g). The conclusion is that there is at most one non-zero term in the sum, and this term is f(x,0)g(0,y). In other words, (f *g)(x,y) = f(x,0)g(0,y). This product f(x,0)g(0,y) is also (fg)(x,y) (element in the x$^{\text{th}}$ row and y$^{\text{th}}$ column of the matrix product fg). Therefore: f *g = fg. We can show in the same way that g*f = fg.

b) $$k(x,y) = (g*h)(x,y) = \sum_{i=0}^{0} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} g(i,j)\, h(x-i,y-j) = \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} g(0,j)\, h(x,y-j)$$

Therefore: $$(f *(g*h))(x,y) = (f *k)(x,y) = \sum_{i=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{j=-0}^{0} f(i,j)\, k(x-i,y-j) = \sum_{i=-\frac{m-1}{2}}^{\frac{m-1}{2}} f(i,0)\, k(x-i,y)$$

$$\overset{\textit{see expression}}{\underset{\textit{of k(x,y) above}}{=}} \sum_{i=-\frac{m-1}{2}}^{\frac{m-1}{2}} f(i,0) \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} g(0,j)\, h(x-i,y-j) = \sum_{i=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} f(i,0)\, g(0,j)\, h(x-i,y-j)$$

$$\overset{\textbf{2/ a)}}{=} \sum_{i=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} (f *g)(i,j)\, h(x-i,y-j) \overset{\textit{definition of}}{\underset{\textit{convolution}}{=}} ((f *g)*h)(x,y)$$

We have shown that f *(g*h) = (f *g)*h.

We proved in class that convolution is bilinear. According to this exercise, it seems that convolution is also commutative and associative. Moreover, f *g = fg when f is an m×1 matrix and g a 1×n matrix.