Recurrence Equations (continure)

- **Linear second-order recurrences with constant coefficients:**
  - A linear second-order recurrence with constant coefficients is of the form

$$ax(n) + bx(n-1) + cx(n-2) = f(n) \tag{1}$$

    where a, b, and c are real numbers and $a \neq 0$.

    * **Second order**: $x(n)$ and $x(n-2)$ are two positions apart in the sequence.
    * **Linear**: The left hand side is a linear combination of the unknown terms of the sequence.
    * **Constant coefficients**: The coefficients are all constants.
    * The equation is **homogeneous** if $f(n) = 0$ for all the n, otherwise it is **inhomogeneous**.

  - This type of recurrences can be solved by neither backward nor forward substitutions.
  - A method for solving this type of recurrences is using characteristic equations.
  - Solving homogeneous recurrence:

$$ax(n) + bx(n-1) + cx(n-2) = 0 \tag{2}$$

    The **characteristic equation** for (2) is quadratic equation:

$$ar^2 + br + c = 0 \tag{3}$$

    The two roots of (2) are

$$r_1 = (-b + \sqrt{b^2 - 4ac})/2a \tag{4}$$

    and

$$r_2 = (-b - \sqrt{b^2 - 4ac})/2a \tag{5}$$

    * **Case 1:** If r_1 and r_1 are real and distinct, the general solution to recurrence (2) is

$$x(n) = \alpha r_1^n + \beta r_2^n \tag{6}$$

      where α and β can be decided by using initial conditions.

* **Case 2:** If r_1 and r_1 are real and equal to each other, i.e. $r_1 = r_2 = r$, the general solution to recurrence (2) is

$$x(n) = \alpha r^n + \beta n r^n \tag{7}$$

  where α and β can be decided by using initial conditions.

* **Case 3:** If r_1 and r_1 are two distinct complex numbers, $r_{1,2} = u \pm iv$, the general solution to recurrence (2) is

$$x(n) = \gamma^n[\alpha \cos n\theta + \beta \sin n\theta] \tag{8}$$

  where α and β can be decided by using initial conditions, $\gamma = \sqrt{u^2 + v^2}$, and $\theta = \arctan u/v$.

– Solving inhomogeneous recurrence:

$$ax(n) + bx(n-1) + cx(n-2) = f(n) \tag{9}$$

* Calculate the particular solution $x(n) = s$. s must satisfy

$$as + bs + cs = f(n) \tag{10}$$

* Calculate the general solution of the corresponding homogeneous recurrence:

$$ax(n) + bx(n-1) + cx(n-2) = 0 \tag{11}$$

* The general solution of (9) is the sum of the two solutions.

– The **general linear kth degree recurrence with constant coefficients**:

$$a_k x(n) + a_{k-1} x(n-1) + ... + a_0 x(n-k) = f(n) \tag{12}$$