

Modeling Arrival Time

The Poisson Distribution



Model Limitations: All Distributions are Uniform



- How do we know that customers will arrive randomly with a Uniform distribution?
 - Actually they probably won't
- How do we know that service time occurs randomly with a Uniform distribution?
 - It probably doesn't
- So we need to model our events with distributions other than Uniform
 - But what distribution should we use?
 - and how can we implement them?
 - C only has rand() which is Uniformly distributed

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Poisson Arrivals



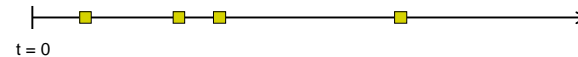
- Poisson Distribution
 - discrete probability distribution
 - describes a random variable representing the number of events that occur in a time interval
- There is a convenient relationship between Poisson (discrete) and Exponential (continuous) distributions
 - Poisson: number of events in a time interval
 - Exponential: time between events
- We exploit these two distributions extensively
 - queueing theory: math is straightforward enough for analysis
 - simulation: "time to next arrival" variates easy to generate

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Poisson Arrivals



- Assuming:
 - arrivals occur one at a time, at some rate
 - arrivals are as likely to occur at any time as at any other
 - nothing else is known about inter-arrival times
- *We can calculate distribution of inter-arrival times*



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Poisson Arrivals (cont.)



- Let Δt represent some small interval of time
- Since we can consider Δt as small as we like, and arrivals are as likely to occur at any instant:
 1. probability of more than one arrival in Δt is 0
 2. probability of one arrival in Δt is proportional to Δt
 - i.e. $\lambda \Delta t$, where λ is a constant reflecting overall arrival rate
 3. arrivals in one time interval are independent of those in another disjoint one
- Let $p(k, t)$ be the probability of k arrivals occurring in the interval $[0, t]$. Consider the interval $[t, t + \Delta t]$:
 - probability of arrival in $[t, t + \Delta t] = \lambda \Delta t$
 - probability of no arrival in $[t, t + \Delta t] = 1 - \lambda \Delta t$

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Poisson Arrivals (cont.)



So:

$$p(k, t + \Delta t) = \text{prob. } k \text{ arrivals in } [0, t] \text{ and zero in } [t, t + \Delta t] \\ + \text{prob. } (k - 1) \text{ arrivals in } [0, t] \text{ and 1 in } [t, t + \Delta t] \\ = p(k, t) \cdot (1 - \lambda \Delta t) + p(k - 1, t) \cdot \lambda \Delta t$$

- Rearranging:

$$\frac{p(k, t + \Delta t) - p(k, t)}{\Delta t} = \lambda [p(k - 1, t) - p(k, t)]$$

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Poisson Arrivals (cont.)



- As $\Delta t \rightarrow 0$, the LHS of this becomes a derivative:

$$\frac{d}{dt} p(k, t) = -\lambda [p(k, t) - p(k - 1, t)]$$
 - this differential equation has initial conditions:
 - $p(k, 0) = 0$ for all $k > 0$
 - $p(0, 0) = 1$
- Solving for this equation and its boundary conditions (which is non-trivial) yields:

$$p(k, t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$
 - probability of k arrivals occurring in $[0, t]$

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Poisson Arrivals (cont.)



- Probability mass function (i.e. $t = 1$):

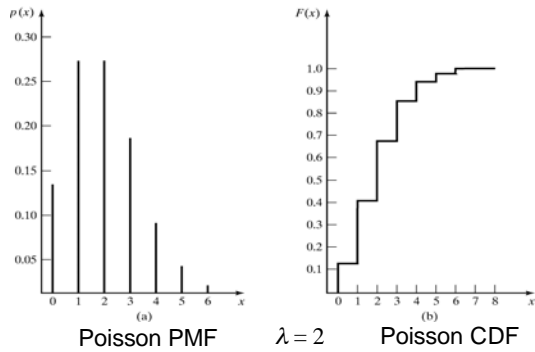
$$p(x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!}, & x = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

- Example (text)
 - computer repair person paged each time there is a service call -- number of beeps per hour is known to agree with a Poisson distribution with mean $\lambda = 2$
 - probability of 3 beeps in next hour?

$$p(3) = \frac{e^{-2} 2^3}{3!} = \frac{(0.135)(8)}{6} = 0.18$$
 - probability of 2 or more beeps?

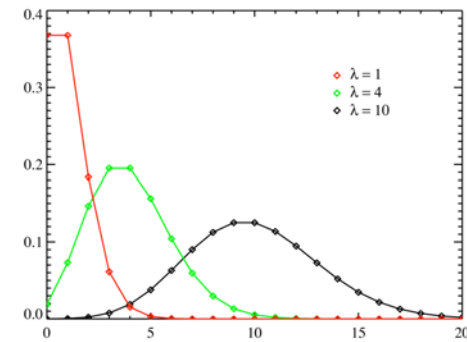
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Poisson Arrivals (cont.)



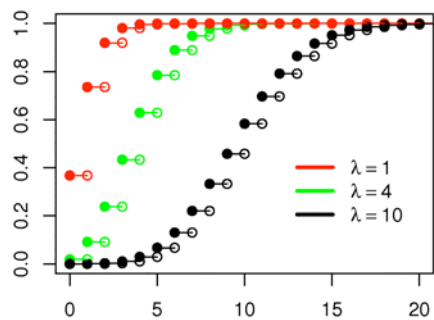
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Poisson Arrivals (cont.)



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Poisson Arrivals (cont.)



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Inter-arrival Time

- Let X be the random variable equal to the length of time between arrivals:

$$F_X(t) = P(X \leq t)$$

- Note: the following two events are complementary (1 and only 1 occurs):
 - waiting time $\leq t$ (i.e. at least one arrival in $[0, t]$)
 - no arrivals in $[0, t]$

- Thus: $F_X(t) + e^{-\lambda t} = 1$

$$F_X(t) = 1 - e^{-\lambda t}$$

$$f_X(t) = \frac{d}{dt} F_X(t) = \lambda e^{-\lambda t} \quad \text{distribution of inter-arrival times is an exponential distribution}$$

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Application to Simulation



- “Poisson Arrivals” commonly used in modelling
 - e.g. particles from a radioactive source have Poisson arrivals
 - in the absence of any information, it is common to assume arrival events are Poisson
- Benefit:
 - inter-arrival time is exponential
 - easily integrated with time-based/event driven simulations
 - relatively simple distribution
 - simple situations can be modelled mathematically when necessary
- Aside
 - service times are often assumed to be exponential, however it is debatable whether this is justified